JANUARY 2010 ANALYSIS QUALIFYING EXAM

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1. Problem 1

(a). Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy and fix $\epsilon > 0$. By uniform continuity, there exists δ such that $\rho(x,y) < \delta$ implies $\sigma(f(x), f(y)) < \epsilon$ for all $x,\ y \in X$. We may choose N such that $m,\ n > N$ implies $\rho(x_n, x_m) < \delta$, so that $\sigma(f(x_n), f(x_m)) < \epsilon$, and $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy as well.

(b). Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy. By completeness, $x_n \to x \in X$, and, by continuity, $f(x_n) \to f(x)$. But then $(f(x_n))_{n\in\mathbb{N}}$ is convergent, hence Cauchy in Y.

2. Problem 2

2.0.1. *(a).* $\mathcal{M}$ is a σ-algebra if

(1) $\varnothing,\ X \in \mathcal{M}$
(2) $U \in \mathcal{M} \implies U^c \in \mathcal{M}$
(3) If $U_1, U_2, \dots \in \mathcal{M}$, then

$$\bigcup_{n=1}^{\infty} U_n \in \mathcal{M}$$

(b). Let $\sigma(C)$ denote the desired σ-algebra. We may define

$$\sigma(C) := \bigcap_{M\supset C} \{M \text{ is a } \sigma - \text{algebra}\}$$

Obviously the intersection of σ-algebras is again a σ-algebra, and this is minimal with respect to inclusion by definition.

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(c). We merely consider countable unions and complements.[1]

If $A \in \mathcal{M}$, then A can be written as $A = \bigcup_{n=1}^\infty C_n$ for $C_n \in \mathcal{C}$. Then,

$$f^{-1}(A) = \bigcup_{n=1}^\infty f^{-1}(C_n) \in \mathcal{M}$$

or, $A = C^c$, $C \in \mathcal{C}$. Then,

$$f^{-1}(A) = f^{-1}(C)^c \in \mathcal{M}$$

So we are done.

3. Problem 3

Define

$$A := \bigcap_{n\geqslant 1} \bigcup_{m \geqslant n} A_m = \limsup_{n\to\infty} A_n$$

As $\nu(\bigcup_{m\geqslant 1} A_m) \leqslant \sum_{m\geqslant 1} 2^{-m} = 1$, we see

$$\begin{aligned}\nu(\bigcap_{n\geqslant 1} \bigcup_{m \geqslant n} A_m) &= \lim_{n\to\infty} \nu(\bigcup_{m\geqslant n} A_m) \\ &\leqslant \lim_{n\to\infty} \sum_{m\geqslant n} 2^{-m} = 0\end{aligned}$$

We also see that

$$\begin{aligned}\mu(\bigcap_{n\geqslant 1} \bigcup_{m \geqslant n} A_m) &= \lim_{n\to\infty} \mu(\bigcup_{m\geqslant n} A_m) \\ &\geqslant \lim_{n\to\infty} \mu(A_n) \\ &= \epsilon\end{aligned}$$

(b). We merely take the contrapositive. Suppose that there exists $\epsilon > 0$ such that for all δ, $\nu(A) < \delta$ but $\mu(A) \geqslant \epsilon$. Then, choose A_n such that $\nu(A_n) < 2^{-n}$ but $\mu(A_n) \geqslant \epsilon$. Setting $A := \bigcap_{n\geqslant 1}\bigcup_{m\geqslant n} A_m$, part (a) shows that

$$\nu(A) = 0, \text{ but } \mu(A) = 0$$

[1]Intuitively, one may think about the σ-algebra generated by a set as doing the "bare minimum" to become a σ-algebra, that is, just take complements and countable unions.

So we are done.

4. Problem 4

Suppose for sake of contradiction that the conclusion is false. Then, $f_n(0) = 1$, but $f_n(x) > \min\{1, 1/x\}$, so that $f_n > \chi_{[0,1]} + \frac{\chi_{(1,\infty)}}{x}$. Integrating this inequality, we find

$$\int_0^\infty f_n(x)dx \geqslant \int_0^1 dx + \int_1^\infty \frac{1}{x}dx = \infty$$

which is a contradiction.

(b). By part (a), $f \leqslant \min\{1, 1/x\}$ as well, so that

$$\begin{aligned}\int_0^\infty |f_n - f| d\mu \leqslant 2^p \int_0^\infty \min\{1, 1/x\} d\mu \\ &= 2^p + 2^p \int_1^\infty \frac{1}{x^p}dx \\ &= 2^p + \frac{2^p}{p-1} = \frac{p2^p}{p-1} < \infty\end{aligned}$$

Then, by Lebesgue's dominated convegence theorem, we may interchange the order of the limit and integral, whence

$$\lim_{n\to\infty} \int_0^\infty |f_n - f| d\mu = 0$$

5. Problem 5

(a). Suppose $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$||fg||_1 \leqslant ||f||_p ||g||_q$$

(b). By Hölder's inequality for $p = 4$, $q = 4/3$,

$$\begin{aligned}\int_0^1 x^{60} f'(x)^{1/4} dx &\leqslant \Big(\int_0^1 x^{80}\Big)^{3/4} \Big(\int_0^1 f'(x)dx\Big)^{1/4} \\ &= \Big(\frac{1}{81}\Big)^{3/4} \cdot (f(1) - f(0))^{1/4} \\ &= \frac{\big(f(1) - f(0)\big)^{1/4}}{27}\end{aligned}$$

For equality to hold, we need that

$$\begin{aligned}
\big(x^{60}\big)^{4/3} &= f'(x) \\
\implies f'(x) &= x^{80} \\
\implies f(x) &= \frac{x^{81}}{81}
\end{aligned}$$

Then, we see that

$$\begin{aligned}
\int_0^1 x^{60} \cdot f'(x)^{1/4} dx &= \int_0^1 x^{80} dx \\
&= \frac{1}{81} \\
&= \frac{\big(f(1) - f(0)\big)^{1/4}}{27}
\end{aligned}$$

6. Problem 6

Note that by absolute continuity, we may write

$$f(y) - f(a) = \int_a^y f'(t)dt$$

Then,

$$\begin{aligned}
(f(y) - f(a))^{n-1} f'(y) &= \Big(\int_a^y f'(t)dt \Big)^{n-1} f'(y) \\
&= \frac{d}{dy} \frac{\Big(\int_a^y f'(t)dt \Big)^n}{n}
\end{aligned}$$

So that

$$\begin{aligned}
\lim_{n\to\infty} \Big(\int_a^x (f(y) - f(a))^{n-1} f'(y) dy \Big)^{1/n} &= \lim_{n\to\infty} \frac{1}{n^{1/n}} \Big(\int_a^x \frac{d}{dy} \Big(\int_a^y f'(t)dt \Big)^n dy \Big)^{1/n} \\
&= \lim_{n\to\infty} \frac{1}{n^{1/n}} \int_a^x f'(t)dt \\
&= f(x) - f(a)
\end{aligned}$$

Which yields the result, as contended.

7. Problem 7

(a). We see:

$$\begin{aligned}||f * g||_1 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)dy \right| dx \\ &\leqslant \int_{\mathbb{R}} |f(y)| \int_{\mathbb{R}} |g(x-y)| dx dy \quad \text{(Fubini-Tonelli)} \\ &= \int_{\mathbb{R}} f(y) dy ||g||_1 \\ &= ||f||_1 \cdot ||g||_1\end{aligned}$$

(b). Note first that the mean value theorem gives that

$$|f(x) - f(y)| \leqslant 2010|x - y|$$

Then, by a simple change of variable, we see

$$(f * g)'(x) = \lim_{h \to 0} \int_{\mathbb{R}} \frac{f(x + h - y) - f(x - y)}{h} \cdot g(y) dy$$

We see that the integrand is bounded by $2010 \cdot |g(y)|$, so that sinc $g \in L^1(\mathbb{R})$, we may employ Lebesgue's dominated convergence theorem to interchange the order of the limit and integral to find

$$\begin{aligned}(f * g)'(x) &= \int_{\mathbb{R}} \lim_{h \to 0} \frac{f(x + h - y) - f(x - y)}{h} \cdot g(y) dy \\ &= \int_{\mathbb{R}} f'(x - y) g(y) dy \\ &= (f' * g)(x)\end{aligned}$$

As asserted.

8. Problem 8

(a). f is holomorphic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b). Note that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= 0\end{aligned}$$

(c). In view of the standard definition of the Wirtinger derivatives, it suffices to show

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \log(|f|) = 0$$

Now, we may write $\log(|f|) = \frac{1}{2}\log(f) + \frac{1}{2}\log(\overline{f})$, so that

$$\begin{aligned}\frac{\partial}{\partial \overline{z}} \log(|f|) &= \frac{1}{2f}\frac{\partial f}{\partial \overline{z}} + \frac{1}{2\overline{f}}\frac{\partial \overline{f}}{\partial \overline{z}} \\ &= \frac{1}{2\overline{f}}\frac{\partial \overline{f}}{\partial \overline{z}}\end{aligned}$$

And, taking $\frac{\partial}{\partial z}$ of the above,

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \log(|f|) &= \frac{-1}{2\overline{f}^2} \cdot \left(\overline{\frac{\partial f}{\partial \overline{z}}}\right)\frac{\partial \overline{f}}{\partial \overline{z}} + \frac{1}{2\overline{f}}\frac{\partial}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} \\ &= \frac{1}{2\overline{f}}\frac{\partial}{\partial \overline{z}}\left(\overline{\frac{\partial f}{\partial \overline{z}}}\right) \\ &= 0\end{aligned}$$

So that $\log(|f|)$ is harmonic, as desired.

9. Problem 9

(a). Suppose that z_0 is a zero of f, so that

$$f(z) = (z - z_0)g(z), \quad g(z_0) \neq 0$$

Then,

$$\begin{aligned}\frac{f'}{f} &= \frac{n(z-z_0)^{n-1}g(z)}{(z-z_0)^n g(z)} + \frac{(z-z_0)^n g'(z)}{(z-z_0)^n g(z)} \\ &= \frac{n}{(z-z_0)} + \frac{g'(z)}{g(z)}\end{aligned}$$

Note that if f has no other zeroes, $\frac{g'(z)}{g(z)}$ is holomorphic, so that

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0$$

by Cauchy's integral theorem. Then, using the above,

$$\int_\gamma \frac{f'}{f} dz = \int_\gamma \frac{n}{z - z_0} = 2\pi i n$$

Now, more generally, suppose

$$f(z) = \prod_{i=1}^{k} (z - z_i)^{n_i} \cdot g(z)$$

where $g(z_i) \neq 0$ for any i. Then,

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{k} \frac{n_i}{z - z_i} + \frac{g'(z)}{g(z)}$$

And, setting $N := \sum_{i=1}^{k} n_i$, we see:

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \cdot N$$

(c). N is simple the sum of the orders of the zeroes of f; that is, we are counting the zeroes of f with multiplicity.

10. Problem 10

(a). False. The Cantor function is the standard counterexample, as $f'(x) = 0$ a.e, f is monotone increasing, yet,

$$f(1) - f(0) = 1 \neq 0 = \int_0^1 f'(x) dx$$

(b). False. Set

$$f_n(x) := \begin{cases} -2n^2 x + 2n, & x \in [0, 1/n] \\ 0, & x \in [1/n, 1] \end{cases}$$

Then, $f_n \to 0$ pointwise a.e, but,

$$\int_0^1 f_n(x) dx = 1 \text{ for all } n \in \mathbb{N}$$

(c). True. We may choose a neighborhood of 0 containing our sequence. Then, by the Casorati-Weierstrass theorem, $f(U\setminus\{0\})$ is dense in $\mathbb{C}$; that is, we may find z_n such that $f(z_n) \to 2010i$ and $z_n \to 0$.